Tutorial 9
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BACK

Questions of today

1. There are some generalizations of Schwarz lemma:
   a. Let f be holomorphic on $\mathbb{D}$ and suppose that $|f(z)| \le M$ for all $z \in \mathbb{D}$. Suppose $\alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{D}$ with $f(z_k) = 0$. Show that

$$|f(z)| \le M \prod_{k=1}^{n} \frac{|z-\alpha_k|}{|1-\overline{\alpha_k} z|}$$

   b. (Schwarz-Pick) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. For $a, b \in \mathbb{D}$, show that

$$\left| \frac{f(a)-f(b)}{1-\overline{f(a)}f(b)} \right| \le \left| \frac{a-b}{1-\bar{a}b} \right|.$$

2. There are some statements making use of the fact that the Caylay transform

$$z \mapsto \frac{z-i}{z+i}$$

   from the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D}$.
   a. Any entire map f with $\mathrm{Re}(f)$ bounded below is constant.
   b. Let $\mathcal{F}$ be a family of function on a region Ω such that the real parts of $\mathcal{F}$ are bounded below, then $\mathcal{F}$ is normal.
   c. (Borel–Carathéodory) Let f be a holomorphic function defined on the closed unit disc $D_R = \{z : |z| \le R\}$. Show that, for $r < R$,

$$\sup_{|z|\le r} |f(z)| \le \frac{2r}{R-r} \sup_{|z|\le R} \mathrm{Re} f + \frac{R+r}{R-r} f(0).$$

3. Suppose $\{f_n\}$ is a sequence of holomorphic functions on Ω, and $f_n \to f$ uniformly on compact subset. Show that f is holomorphic and $f_n^{(k)} \to f^{(k)}$ for any positive integral k.
4. (Hurwitz) Let Ω be a region (so it is connected), and $\{f_n\}$ a sequence of holomorphic functions on Ω. Suppose $f_n \to f$ uniformly on compact subsets, show that either $f \equiv 0$ or f is nowhere vanishing.
5. (Vitali) Let $\{f_n\}$ be a locally bounded sequence of holomorphic functions on Ω, and f is a holomorphic function on Ω. Suppose the set $A = \{z \in \Omega : f_n(z) \to f(z)\}$ has a limit point in Ω, show that $f_n \to f$ uniformly on compact subsets.

Hints & solutions of today
